2 Exact and Approximate Evaluation of Sums and Integrals

Definition 2.1 *An asymptotic sequence is a set of functions* $\{\phi_n(z)\}$ *such that*

$$\phi_{n+1}(z) = o(\phi_n(z)) \quad , \quad as \quad z \to z_0 \ . \tag{1}$$

(Usually we take $\phi_n = z^{-n}$*, and* $z_0 = \infty$*).*

Definition 2.2 *If* $\{\phi_n(z)\}$ *is an asymptotic sequence, then the asymptotic expansion for a function* $f(z)$ *is*

$$f(z) \sim \sum_{r=0}^{\infty} a_r \phi_r(z) \ , \tag{2}$$

provided that

$$\left| f(z) - \sum_{r=0}^{n-1} a_r \phi_r(z) \right| = O(\phi_n) \quad or \quad o(\phi_{n-1}) \ , \tag{3}$$

as $z \to z_0$*. (i.e. the remainder after* n *terms is smaller than the last included term, or the same order as the first neglected term)*

Some important properties of asymptotic expansions are (Here consider $f \sim \sum a_r z^{-r}$ always.):

1. Asymptotic expansions depend on the sector (i.e $\arg(z)$). For example,

   $$e^{-z} + \frac{1}{z} \sim \frac{1}{z} \quad , \quad |\arg(z)| < \frac{\pi}{2} \ . \tag{4}$$

   (no more terms, since e^{-z} is smaller than any power of z). But,

   $$e^{-z} + \frac{1}{z} \not\sim \frac{1}{z} \quad , \quad \frac{3\pi}{2} < |\arg(z)| < \frac{\pi}{2} \ . \tag{5}$$

   If the asymptotic expansion of $f(z)$ is different in different sectors, we say it exhibits *Stokes' phenomenon.*

   Theorem 2.1 *If* $f(z)$ *is single-valued and holomorphic for* $|z| \geq a$*, and*

   $$f(z) \sim \sum_{r=0}^{\infty} a_r z^{-r} \ , \tag{6}$$

   is valid for all $\arg(z)$ *(i.e., doesn't exhibit Stokes' phenomenon), then the series is in fact convergent; i.e.*

   $$f(z) = \sum_{r=0}^{\infty} a_r z^{-r} \ . \tag{7}$$

Proof 2.2 *f is single-valued, holomorphic for $|z| \geq a$, therefore $f(z) = \sum_{-\infty}^{\infty} c_n z^n$, with*

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} \, dz \; . \tag{8}$$

Choose C to be a large circle, radius R. Then

$$|c_n| \leq \frac{1}{2\pi} \left| \int_0^{2\pi} f(Re^{i\theta}) \, d\theta \right| \frac{1}{R^n} \; . \tag{9}$$

Now, since $f(z) \sim \sum_0^{\infty} a_r z^{-r}$, $f \to a_0$ as $|z| \to \infty$. Therefore, we can find M such that $|f| < M$ for large enough $|z|$. This implies that

$$|c_n| < \frac{M}{R^n} \quad , \quad (n > 0) \; . \tag{10}$$

But R can be as large as we like, so $c_n = 0$ for $n > 0$. Also, $a_n = c_{-n}$, since asymptotic expansions are unique (see next property).

2. For a given range of $\arg(z)$, the asymptotic expansion of $f(z)$ is unique

   To see this, let $f(z) \sim \sum_0^{\infty} a_n z^{-n}$ as $z \to \infty$ in a given sector. Then $f \sim a_0$ as $z \to \infty$, and

   $$(f - a_0)z \to a_1 \; , \tag{11}$$

   as $z \to \infty$. Similarly

   $$\left(f - \sum_0^{n-1} a_r z^{-r} \right) z^n \to a_n \; , \tag{12}$$

   as $z \to \infty$. Thus, the coefficients $\{a_n\}$ are uniquely defined. (Note that the converse does not hold).

3. Asymptotic expansions can be added and multiplied as if they were convergent

Let's now see how we might calculate asymptotic expansions for several different classes of functions.

2.1 Watson's Lemma and Laplace's Method

Lemma 2.1 (Watson) *Let*

$$F(z) = \int_0^\infty e^{-zt}\phi(t)\,dt \quad , \quad \Re(z) \geq \delta > 0 \ , \tag{13}$$

with $\phi = \sum_0^\infty b_n t^n$ *for* $|t| < R$. *Then*

$$F(z) \sim \sum_0^\infty \frac{b_n n!}{z^{n+1}} \ . \tag{14}$$

It is important to note here that the right hand side is merely the left hand side expanded and integrated term by term. However, it is the fact that the result is an asymptotic expansion that is nontrivial. This is because the summation need not converge uniformly in t for all t in the range of integration. Thus, it is not clear that we can interchange the order of integration and summation.

Under more restrictive circumstances we could just integrate by parts to show this. However, Watson's lemma works in more general situations, and a more subtle proof is required. I won't give the proof here, although if we have time I may come back and supply it later.

Laplace's method is a way to calculate asymptotic expansions for functions of the form

$$F(x) = \int_a^b e^{xh(u)} g(u)\,du \ , \tag{15}$$

as $x \to +\infty$ (x real).

The rough argument is that the largest contribution comes from the biggest value of $h(u)$, say $h(u_0)$, which is exponentially larger than any other contribution. We'll see how this works in 2 distinct situations. In both these, Watson's lemma is crucial to obtaining the final result.

1. $\underline{h'(u_0) = 0 \text{ ; (a calculus-type maximum)}}$

   Begin by taking Taylor series of h and f about u_0:

$$\begin{aligned} F(x) &= \int_a^b \exp\left\{x\left[h(u_0) + \frac{1}{2}(u-u_0)^2 h''(u_0) + \cdots\right]\right\}[g(u_0) + (u-u_0)g'(u_0) + \cdots]\,du \\ &\sim e^{xh(u_0)} \int_{-\infty}^{\infty} \exp\left[\frac{1}{2}x\tau^2 h''(u_0)\right][g(u_0) + \cdots]\,d\tau \ , \end{aligned} \tag{16}$$

where $\tau = u - u_0$ and we can extend the range of integration to $(-\infty, \infty)$ since any extra contributions are negligible (the dominant contribution comes from $\tau = 0$)

Now integrate term by term using Watson's lemma, to obtain

$$F(x) \sim e^{xh(u_0)} \left[g(u_0) \sqrt{\frac{2\pi}{-xh''(u_0)}} + O(x^{-3/2}) \right] . \tag{17}$$

2. <u>$h'(u_0) \neq 0$</u>

   In this case we have $u_0 = b$ (or a). Now a Taylor expansion about u_0 yields

$$\begin{aligned} F(x) &\sim e^{xh(u_0)} g(u_0) \int_{-\infty}^{0} e^{x\tau h'(u_0)}\, d\tau \\ &\sim e^{xh(u_0)} g(u_0) \frac{1}{xh'(u_0)} + O(x^{-2}) . \end{aligned} \tag{18}$$

Let's see immediately how this works by applying what we've just learned to an example that is well-known (the result at least) to some of you.

Consider expanding $\Gamma(x+1)$ as $x \to \infty$, for x real. If you know what the Γ-function is, you'll know that the answer we hope to get is known as *Stirling's formula*, and is very useful in all types of situations in physics. If you haven't heard of the Γ-function, then this will still be a good example of how to use Laplace's method.

The Γ-function has an integral expression given by

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x \, dt . \tag{19}$$

Although this appears to already be in the correct form to apply Laplace's method to, we must transform it because the largest value of the exponential occurs at $t = 0$, where t^x vanishes. Therefore, we'll write

$$\begin{aligned} \Gamma(x+1) &= \int_0^\infty e^{-t + x\log(t)}\, dt \\ &= x^{x+1} \int_0^\infty e^{x(-u+\log(u))}\, du , \end{aligned} \tag{20}$$

where we have made the change of variables $t = xu$ not because it is essential, but because it makes things neater because the position of the maximum stays at a fixed point and doesn't go to infinity as we take the asymptotic limit.

Now, $h(u) = -u + \log(u)$ has a maximum at $u = 1$. We will only be interested in this example in getting the leading term of the expansion. Therefore, we Taylor expand everything about $u = 1$ as far as the first non-constant term. This gives

$$\begin{aligned}\Gamma(x+1) &= x^{x+1}\int_0^\infty \exp\left\{x\left[h(1)+\frac{1}{2}(u-1)^2h''(1)+\cdots\right]\right\}du \\ &\sim x^{x+1}e^{-x}\int_{-1}^\infty e^{xs^2/2}\,ds\ ,\end{aligned} \tag{21}$$

with $s = u - 1$. The $\sim$ here comes from Watson's lemma. We could have expanded to higher order but have chosen not to. We can extend the limit of integration since any contribution from the range $(-\infty, -1)$ is subdominant. Thus, to leading order we obtain

$$\Gamma(x+1) \sim x^{x+1}e^{-x}\left(\frac{2\pi}{x}\right)^{1/2}\ , \tag{22}$$

which you may recognize as the leading term in Stirling's formula.

2.2 Riemann-Lebesgue Lemma and Method of Stationary Phase

Lemma 2.2 (Riemann-Lebesgue) *Let $q(t)$ be piecewise continuous on the compact interval $[a, b]$. Then, for real x*

$$I(x) \equiv \int_a^b e^{ixt}q(t)\,dt = o(1)\ , \qquad \text{as} \quad x \to \infty\ . \tag{23}$$

Proof 2.3 *Assume w.l.o.g that $q(t)$ is continuous on $[a, b]$ so that for any given $\epsilon > 0$, the interval $[a, b]$ can be divided into $n - 1$ subintervals in each of which $q(t)$ varies by less than 2ϵ. Then, $\exists\{t_n\}$ such that $a = t_0 < t_1 < \ldots < t_n = b$, with $|q(t) - q(t_k)| < \epsilon$, for $t \in [t_{k-1}, t_k]$. Also, $q(t)$ is bounded in $[a, b]$, so $\exists Q$ such that $|q(t)| < Q\ \forall\ t \in [a, b]$. Then*

$$I(x) = \sum_1^n q(t_i)\int_{t_{i-1}}^{t_i} e^{ixt}\,dt + \sum_1^n q(t_i)\int_{t_{i-1}}^{t_i}[q(t)-q(t_i)]e^{ixt}\,dt\ . \tag{24}$$

Now,

$$\left|\int_{t_{i-1}}^{t_i} e^{ixt}\,dt\right| = \left|\frac{e^{ixt_i}-e^{ixt_{i-1}}}{ix}\right| \le \frac{2}{x}\ , \tag{25}$$

and

$$\left|\int_{t_{i-1}}^{t_i}[q(t)-q(t_i)]e^{ixt}\,dt\right| \le \epsilon(t_i - t_{i-1})\ . \tag{26}$$

Putting these together, we obtain

$$|I(x)| \le Q\frac{2}{x}n + \epsilon(b-a)\ , \tag{27}$$

which can be made as small as you like by choosing ϵ small enough and/or choosing x large enough.

The method of stationary phase is a way to calculate asymptotic expansions for functions of the form

$$I(x) = \int_a^b e^{ixh(u)} g(u)\, du \ . \tag{28}$$

(with h twice differentiable and g once differentiable) as $x \to +\infty$ (x real).

The rough argument is that the largest contribution comes from the place where the integrand oscillates least, since where rapid oscillations occur, one expects cancellations to occur. More formally, making the substitution $h(u) = t$, and using the Riemann-Lebesgue lemma, we see that the above expression is $o(1)$ unless there's a place where $h' = 0$.

2.3 The Saddle-Point Method

This technique is used to find the asymptotic form of general functions of the type

$$I(z) = \int_A^B e^{zh(t)} g(t)\, dt \ , \tag{29}$$

where A and B are complex in general, and h is holomorphic, as $z \to \infty$.

Writing $h = \phi + i\psi$, note that if h is not holomorphic, the dominant contributions are hard to find. The Laplace contributions from the maximum value of ϕ may be cancelled by rapid oscillations of ψ, and vice-versa. However, as we'll see, for holomorphic h, the Cauchy-Riemann equations help us.

Note also in passing that the magnitude of e^{zh} is determined by ϕ via $|e^{zh}| = e^{z\phi}$.

2.3.1 Notes on Paths of Constant $\Im(h) = \psi$

Paths of constant ψ are paths along which ϕ increases/decreases most rapidly: To see this, note that the Cauchy-Riemann equations give

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad , \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \ , \tag{30}$$

and therefore

$$\nabla\phi . \nabla\psi = 0 \ . \tag{31}$$

Now, $\nabla\phi$ is normal to the lines of constant ϕ, so paths of constant ϕ are orthogonal to paths of constant ψ. These paths form a grid over a region where $h(t)$ is holomorphic. Also, paths of constant ψ are paths of maximum descent/ascent of ϕ.

The biggest value of ϕ along a path of constant ψ occurs at a saddle point of ϕ (or at the endpoint). To see this, let γ be any path from A to B. Assume $\phi(t) > \phi(A)$, and $\phi(t) > \phi(B)$ for some t on γ.

Let $\phi(t_0)$ be the biggest value of ϕ on γ. Then $\nabla\phi.\mathbf{b} = 0$ at t_0, where $\mathbf{b}$ is tangent to γ. If γ is a path of constant ψ, then $\nabla\psi.\mathbf{b} = 0$ on γ. Together with the Cauchy-Riemann equations, these imply $\nabla\phi = \nabla\psi = 0$ at t_0, i.e. $h'(t_0) = 0$.

Thus, the stationary phase point and the Laplace points coincide. Therefore, t_0 is a *saddle point* of ϕ because the C-R equations imply $\nabla^2\phi = 0$, which means that the eigenvalues must have opposite signs.

2.3.2 Nature of Paths of ψ = constant near a General Point

In general (i.e. $h' \neq 0$) there is exactly one path of constant ψ through each point t_0:

Near t_0, we can write $h(t) = h(t_0) + (t - t_0)h'(t_0) + \ldots$. Further, write $h'(t_0) = |h'(t_0)|e^{i\alpha}$, and $(t - t_0) = re^{i\theta}$. The path of constant $\Im(h)$ near t_0 is given by

$$\Im[(t - t_0)h'(t_0)] = 0 \ . \tag{32}$$

So $\Im(h(t)) = \Im(h(t_0))$. i.e. $|h'(t_0)|r\sin(\alpha + \theta) = 0$, and thus $\theta = -\alpha$. Hence, there is only one path of constant ψ, at angle $-\alpha$ ($\pi + \alpha$ is the same path).

2.3.3 Nature of Paths of ψ = constant near Saddle Points

If $h'(t_0) = 0$, write $h(t) = h(t_0) + (t - t_0)^2 h''(t_0)/2 + \ldots$, near t_0. Further, write $h''(t_0) = |h''(t_0)|e^{i\alpha}$, and $(t - t_0) = re^{i\theta}$. Then, as above, we obtain $\sin(2\theta + \alpha) = 0$, which implies

$$\theta = -\alpha/2 \ , \qquad \text{or} \quad \theta = -\alpha/2 + \pi/2 \ . \tag{33}$$

So the two paths are at right angles. One is the steepest ascent of ϕ, the other is the steepest descent of ϕ. (Note, if $h''(t_0) = 0$ also, get three paths).

2.3.4 Method in Theory

1. If there exists a path γ, from A to B of constant ψ, then let γ be given by $t = t(s)$, for real s. Then

$$\int_A^B e^{zh(t)} g(t)\, dt = \int_\gamma \left(e^{z\phi} g(t) \frac{dt}{ds} \right) ds e^{i\psi(A)z} \ , \tag{34}$$

which can be evaluated by Laplace's method. If the original path of integration has to be deformed onto γ, there may be additional contributions due to, for example, poles between the two paths.

2. If there exists a path γ, from A to B of constant ϕ, then use the method of stationary phase.

3. In general, suppose $\phi(A) \geq \phi(B)$. Try to find a path from A to B consisting of curves of constant ϕ and curves of constant ψ.

2.3.5 Method in Practice

1. Find the saddle points (i.e. $h' = 0$)

2. Use the saddle point with largest $\Re(h)$

3. Find the steepest *descent* path across the saddle:

$$\begin{aligned} h(t) &= h(t_0) + (t - t_0)^2 h''(t_0)/2 + \dots \ , \\ h''(t_0) &= |h''(t_0)| e^{i\alpha}, and (t - t_0) = r e^{i\theta} \ , \\ \phi(t) &= \phi(t_0) + |h''(t_0)| r^2 \cos(\alpha + 2\theta)/2 + \dots \ . \end{aligned} \tag{35}$$

   Then, maximum descent is when $\cos(\alpha + 2\theta) = -1$, i.e. for

$$\theta = \left(\frac{\pi - \alpha}{2} \right) \ . \tag{36}$$

4. Taylor expand and do the Gaussian integral:

$$\begin{aligned} \int_A^B e^{zh(t)} g(t)\, dt &= g(t_0) e^{zh(t_0)} \int_{-\infty}^{\infty} e^{zh''(t_0)(t-t_0)^2/2} \frac{dt}{dr}\, dr + \cdots \\ &\simeq g(t_0) e^{zh(t_0)} \int_{-\infty}^{\infty} e^{-z|h''(t_0)|r^2/2} e^{i\theta}\, dr \\ &\sim g(t_0) e^{zh(t_0)} \sqrt{\frac{2\pi}{z|h''(t_0)|}} e^{i(\pi-\alpha)/2} \ . \end{aligned} \tag{37}$$

Let's see how all this works in practice with an example. Consider the Bessel-type function $H_\nu^{(2)}(x)$, defined by

$$H_\nu^{(2)}(x) = -\frac{1}{\pi} \int_{-i\infty}^{\pi+i\infty} e^{i(\nu t - x\sin(t))}\, dt \ , \tag{38}$$

where x and ν are real. We wish to find the asymptotic behavior in the limit $x \to \infty$ and $\nu \to \infty$ in the case $x < \nu$.

The answer depends critically on the relative magnitudes of x and ν, and it is convenient to set $x = \nu/\cosh(\beta)$, with $\beta > 0$, to deal with the case $x < \nu$. We obtain

$$H_\nu^{(2)}\left(\frac{\nu}{\cosh(\beta)}\right) = -\frac{1}{\pi} \int_{-i\infty}^{\pi+i\infty} e^{i\nu(t - \sin t/\cosh\beta)}\, dt \ . \tag{39}$$

Now, let

$$h(t) = i\left(t - \frac{\sin t}{\cosh\beta}\right) \ . \tag{40}$$

The saddle points occur where $h'(t) = 0$, i.e. when $\cos t = \cosh\beta$, so there are saddle points at $t = \pm i\beta + 2n\pi$, for all integers n. The ones most relevant to the path of integration are at $\pm i\beta$. To see this, sketch the paths of constant $\Im(h)$ and verify that, if one kept on such paths, one could get from $-i\infty$ to $\pi + i\infty$ without crossing any other saddle points. First let us see which saddle point gives the dominant contribution.

Near $\pm i\beta$, we have

$$\begin{aligned} h(t) &= h(\pm i\beta) + \frac{1}{2!}(\pm i\beta - t)^2 h''(\pm i\beta) + \cdots \\ &= \pm(-\beta + \tanh\beta) + \frac{1}{2!}(\pm i\beta - t)^2(\mp\tanh\beta) + \cdots \ . \end{aligned} \tag{41}$$

Close to the saddle point, the magnitude of the integrand is determined by the first (i.e. the constant) term in the Taylor series. Since $\beta > \tanh\beta$, the contribution from $t = -i\beta$ is exponentially larger than the contribution from $+i\beta$. We would therefore expect the steepest descent paths to cross straight over the saddle at $t = -i\beta$, but to turn through a right-angle at $t = +i\beta$, so that $\Re[h(t)]$ continues to decrease.

The steepest descent paths are given by

$$\Im\left[i\left(t - \frac{\sin t}{\cosh t}\right)\right] = \text{ constant } , \tag{42}$$

and for a path passing through t_0, the value of this constant is $\Im[i(t_0 - \sin t_0/\cosh t_0)]$, which vanishes in the case of $t_0 = \pm i\beta$. The steepest descent paths through $\pm i\beta$ are therefore given by

$$x\cosh\beta - \sin x \cosh y = 0 \ , \tag{43}$$

where $t = x + iy$. Clearly, one path is $x = 0$, but the other one can only be sketched roughly. It is a good idea to know roughly where these paths go, but it is not necessary to know them exactly, unless: either there are singularities in the integrand, so that there may be extra contributions from deforming the contour of integration to the path of constant $\Im[h]$ (e.g. a pole contribution); or further terms in the asymptotic expansion are required.

The required asymptotic behavior is the Laplace contribution from the path which crosses the dominant saddle in the direction corresponding to steepest descent. This direction can be found as follows. Near t_0 we write $(t - t_0) = re^{i\theta}$, then the direction of paths of constant $\Im[h]$ is given by $\sin(\alpha + 2\theta) = 0$, where α is the phase of $h''(t_0)$. To find which of these is the steepest descent path, look at the real part of h near t_0:

$$\Re[h(t)] = \Re[h(t_0)] + \frac{1}{2}r^2|h''(t_0)|\cos(\alpha + 2\theta) \ , \tag{44}$$

so $\Re[h]$ decreases fastest when $\cos(\alpha + 2\theta) = -1$; i.e. when $\theta = (\pi - \alpha)/2$. You will probably find it easiest to do the calculations afresh each time, rather than try to remember this formula.

Near $t = -i\beta$

$$h(t) = (\beta - \tanh\beta) + \frac{1}{2}(-i\beta - t)^2 \tanh\beta + \cdots \ , \tag{45}$$

so that the path $t = -i\beta + ir$ (r real) corresponds to steepest descent.

The saddle point contribution is

$$\begin{aligned} H_\nu^{(2)}\left(\frac{\nu}{\cosh\beta}\right) &\sim -\frac{1}{\pi}e^{\nu(\beta - \tanh\beta)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}r^2\nu\tanh\beta} i\, dr \\ &\sim -\frac{i}{\pi}e^{\nu(\beta-\tanh\beta)}\sqrt{\frac{2\pi}{\nu\tanh\beta}} \ . \end{aligned} \tag{46}$$

Note that we are guaranteed that this is the leading term in the asymptotic expansion by Watson's lemma.